Optimal Virtualized In-Network Processing with Applications to Aggregation and Multicast

KuVS Best Master Thesis Prize 2014

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NetSys 2015, Cottbus
March 11th, 2015

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Mindset: Service Deployment

Setting: Network Virtualization, e.g. SDN + NFV

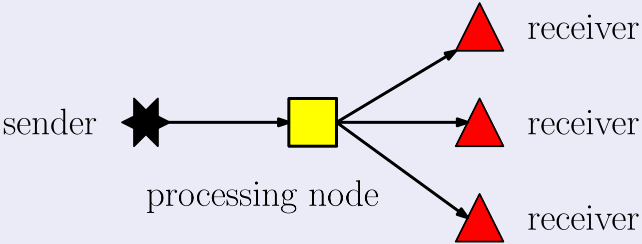
- Routes can be flexibly established on a per flow basis
- Functions can be flexibly placed on specific nodes

Task: Service Deployment

- Given: ‘communication service’ shall be established
- Task: Find an *optimal* virtual topology *and* an embedding of the service on the physical network

Communication Schemes: Multicast (same old! same old?)

processing = duplication + reroute



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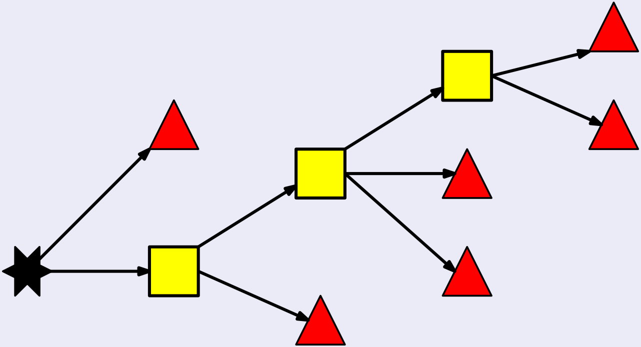
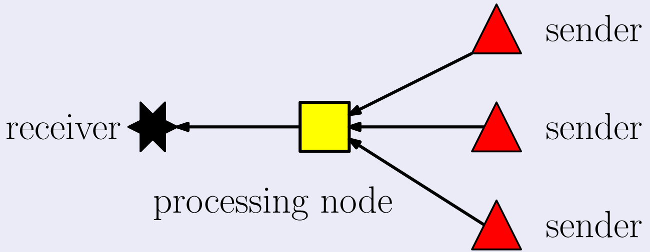


Figure: Hierarchy of processing nodes

Communication Schemes: Aggregation

processing = merge + reroute



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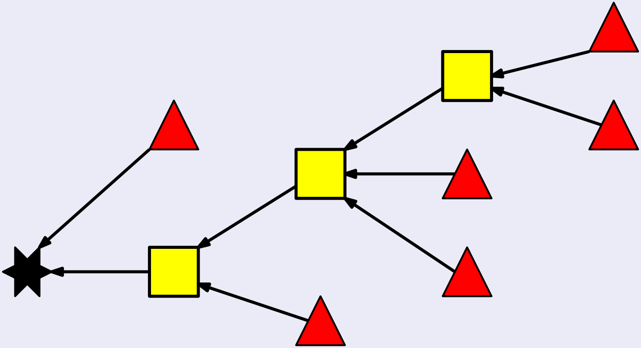


Figure: Hierarchy of processing nodes

Introductory Example

Introductory Example

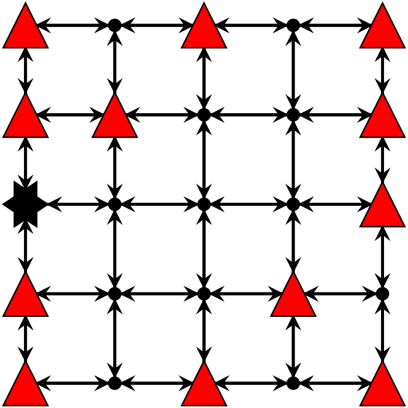
Aggregation scenario

grid graph: 14 senders, one receiver

Virtualized links

data can be routed arbitrarily

receiver sender



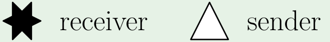
Without in-network processing: Unicast

Solution Method

- minimal cost flow

Solution uses

- 41 edges
- 0 processing nodes



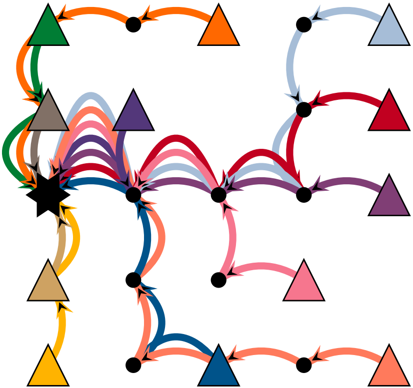


Figure: Unicast solution

With in-network processing at all nodes

Solution Method

- Steiner arborescence

Solution uses

- 16 edges
- 9 processing nodes

receiver	sender
processing node	sender with processing

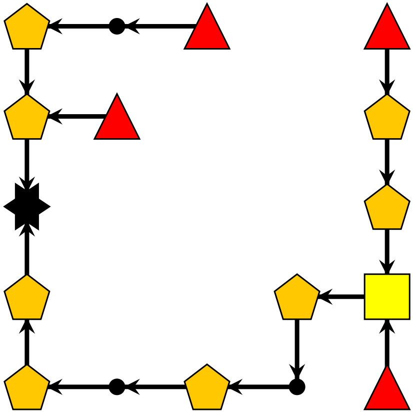
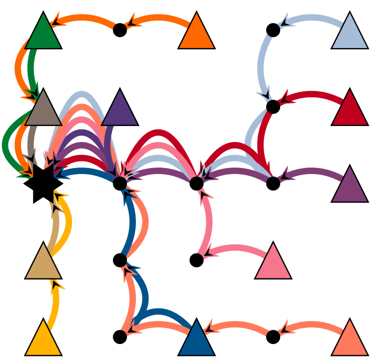
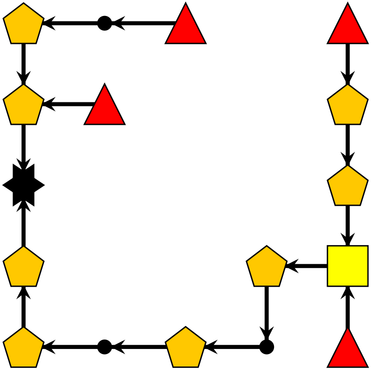


Figure: Aggregation tree

How to Trade-off?



vs.



What we aim for

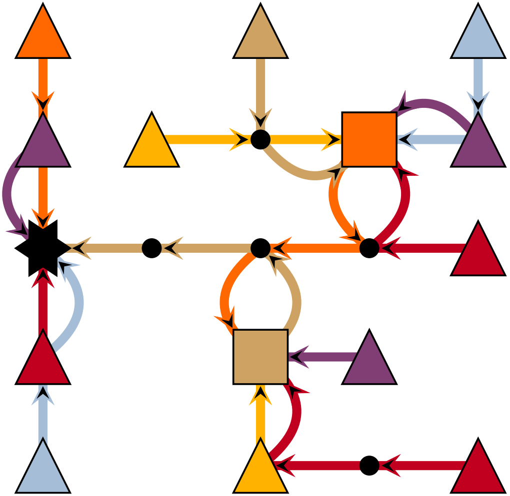
Solution uses

- 26 edges
- 2 processing nodes

receiver

sender

processing node



Solution Structure

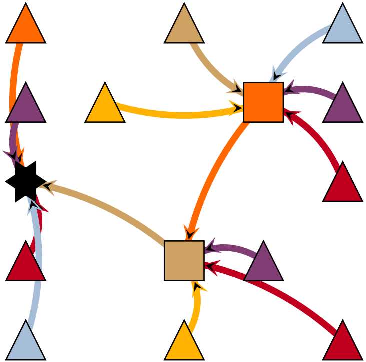


Figure: Virtual Arborescence

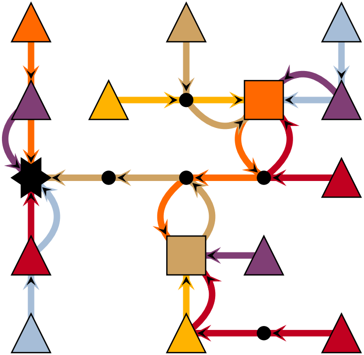


Figure: underlying routes

New Model: Constrained Virtual Steiner Arborescence Problem

Definition: CVSAP (Aggregation Variant)

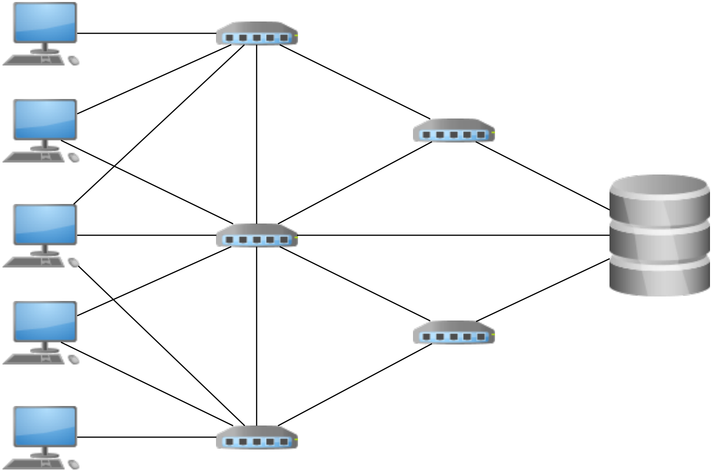
Find a Virtual Arborescence connecting senders to the single receiver, s.t.

1. bandwidth of substrate is not exceeded,
2. inner nodes are capable of processing flow,
3. the processing nodes' capacities are not exceeded,

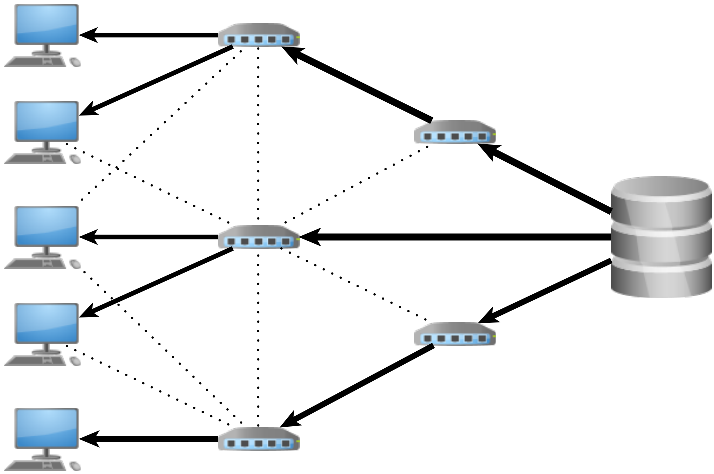
<u>minimizing</u> the joint cost for bandwidth allocations and function placement.

Applications

Service Replication

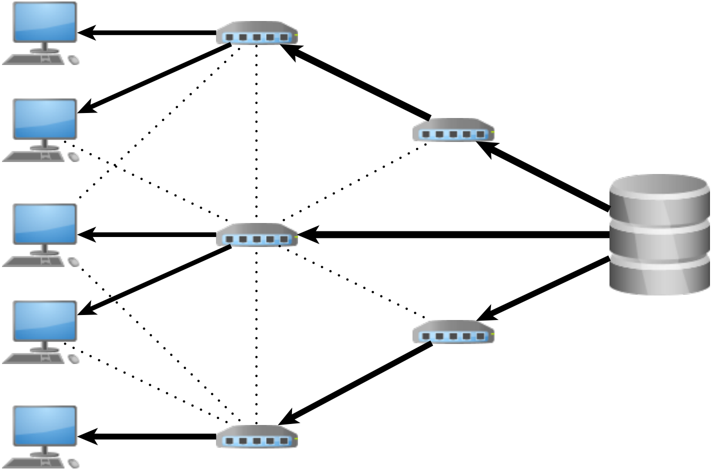


Service Replication

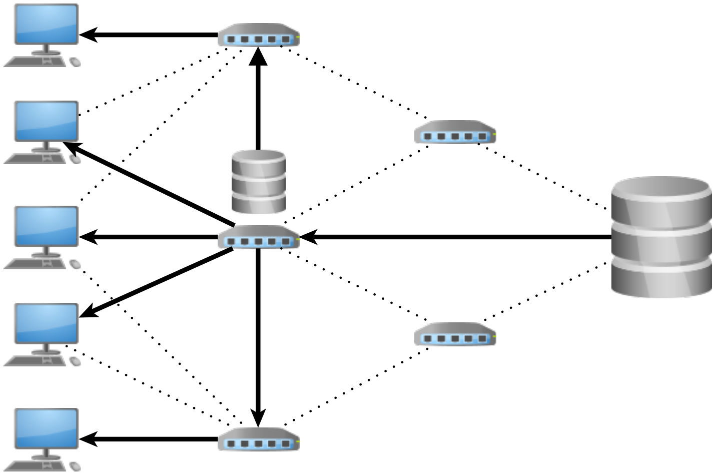


Service Replication

What if backend links are congested?

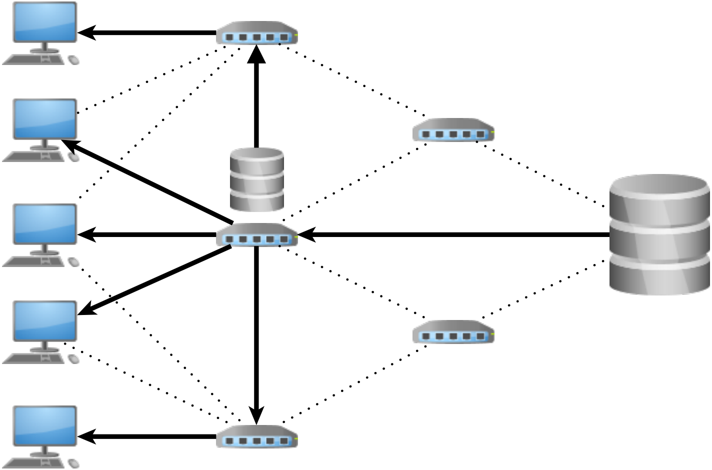


Service Replication

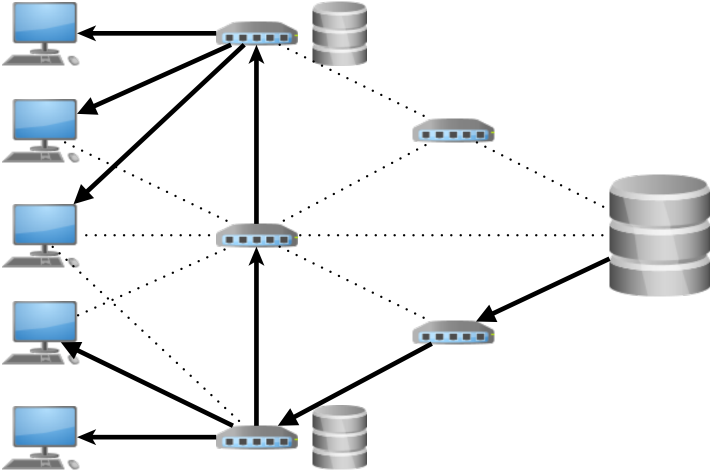


Service Replication

What if only '3' users can be handled?



Service Replication



Applications

	Network	Application	Technology, e.g.
multicast	ISP	service replication / cache placement [10, 11]	middleboxes / NFV + SDN
	backbone	optical multicast [6]	ROADM + SDH
	all	application-level multicast [15]	different
aggregation	sensor network	value & message aggregation [5, 8]	source routing
	ISP	network analytics: Gigascope [3]	middleboxes / NFV + SDN
	data center	big data / map-reduce: Camdoop [2]	SDN

edge capacities

processing node locations

processing node capacities

edge costs

costs for installing processing functionality

Results

Solution Approaches

Wishful thinking: there exists a

- polynomial time algorithm
- solving CVSAP to optimality
- considering all constraints.

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Theorem: Inapproximability of CVSAP

Finding a feasible solution is NP-complete!

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Theorem: Inapproximability of CVSAP

Finding a feasible solution is NP-complete!

Approximations

- polynomial
- quality guarantee
- weaker models

Exact Algorithms

- non-polynomial
- optimality
- full model

Heuristics

- polynomial
- no solution guarantee
- full model

Thesis' core: comprehensive algorithmic study

Algorithms

Approximations
- NVSTP
- VSTP
- VSAP

Exact Algorithms
- multi-commodity flow
- single-commodity flow
  $\rightarrow$ VirtuCast

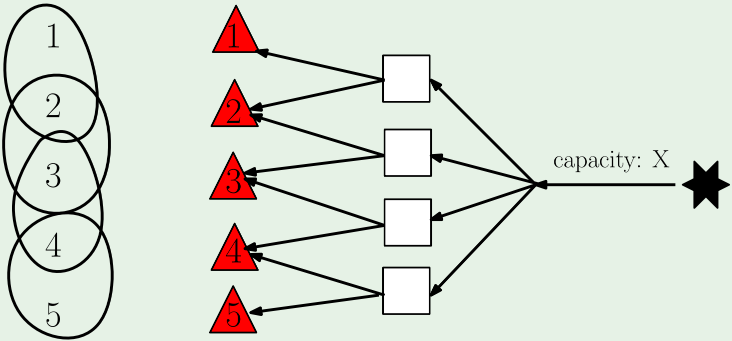
LP-based Heuristics
- FlowDecoRound
- MultipleShots
- GreedyDiving

Combinatorial Heuristic
- GreedySelect

Inapproximability

Inapproximability

Reduction from Set Cover: Does a set cover of size X exist?

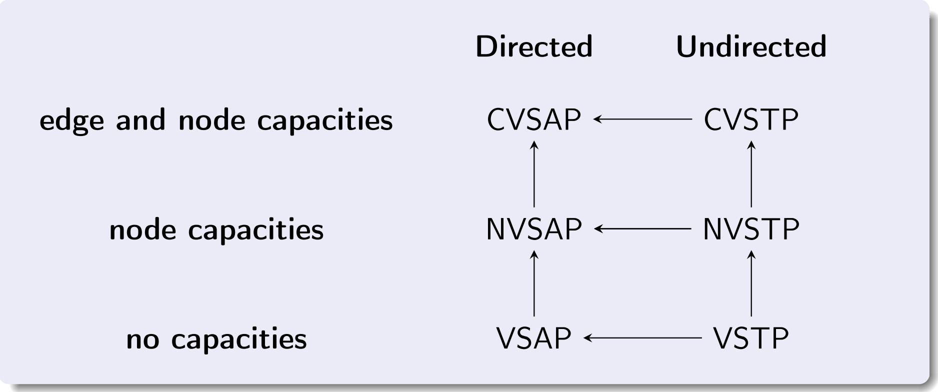


Theorem:

Finding *a feasible* solution is already NP-complete.

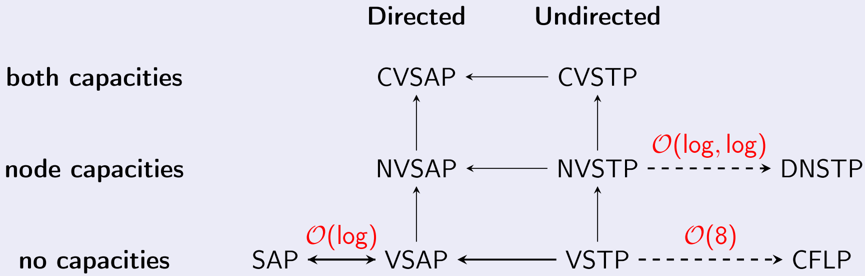
Approximation Algorithms for Variants

Variants


Directed
Undirected
edge and node capacities
CVSAP
CVSTP
node capacities
NVSAP
NVSTP
no capacities
VSAP
VSTP

Approximation via related problems

Results



Bottom Line

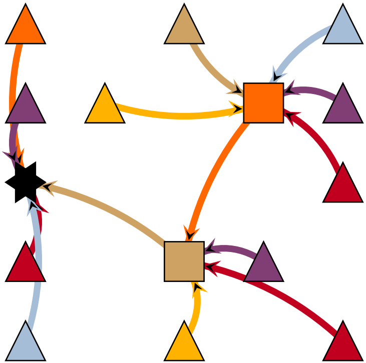
- Better understanding of the problems core complexity: virtualized links & restricted network function placement
- Obtained lower bounds and approximations

Exact Algorithms for CVSAP

Multi-Commodity Flow (MCF) Integer Program

First approach: MCF IP

- 'the simple way to do it'
- explicitly represent virtual arborescence



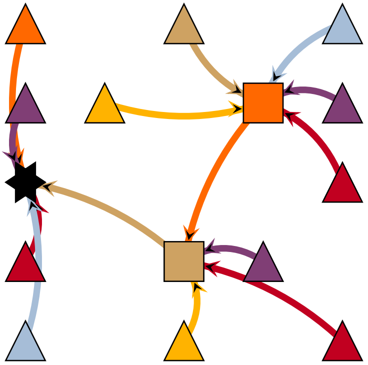
Multi-Commodity Flow (MCF) Integer Program

First approach: MCF IP

- 'the simple way to do it'
- explicitly represent virtual arborescence

Does not scale well

- number of binary variables: $\geq$ #processing nodes $\cdot$ #edges



Single-Commodity Flow IP

Single-commodity flow formulation

- computes *aggregated* flow on edges independently of the origin
- does not represent virtual arborescence

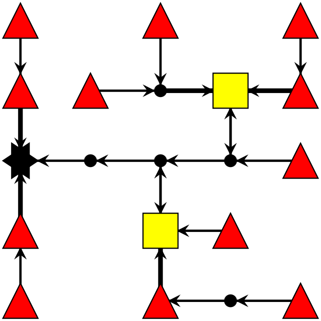


Figure: Single-commodity

Multi- vs Single-Commodity

Example: 6000 edges and 200 Steiner sites

- Single-commodity: 6000 integer variables
- Multi-commodity: 1,200,000 binary variables

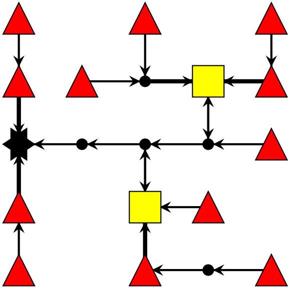


Figure: Single-commodity

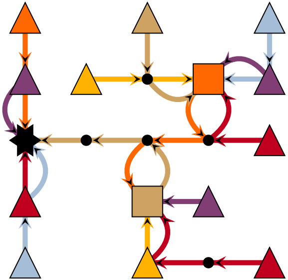


Figure: Multi-commodity

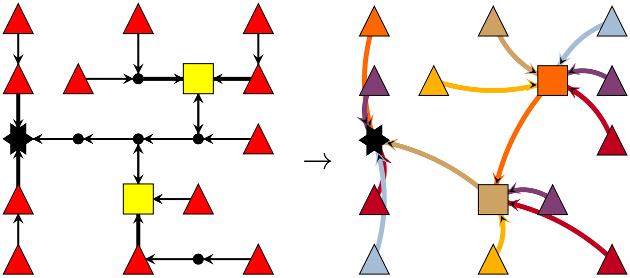
VirtuCast Algorithm

VirtuCast Algorithm

Outline of VirtuCast

1. Solve single-commodity flow IP formulation.
2. Decompose IP solution into Virtual Arborescence.

How to decompose?



(a) IP solution (b) Virtual Arborescence

Complete Formulation

$$
\begin{aligned}
\text{minimize} \quad & C_{\mathrm{IP}}(x,f) = \sum_{e \in E_G} \mathbf{c}_e f_e + \sum_{s \in S} \mathbf{c}_s x_s \\
\text{subject to} \quad & f(\delta^+_{E_{\mathrm{ext}}}(v)) = f(\delta^-_{E_{\mathrm{ext}}}(v)) && \forall\, v \in V_G \\
& f(\delta^+_{E^R_{\mathrm{ext}}}(W)) \geq x_s && \forall\, W \subseteq V_G, s \in W \cap S \neq \emptyset \\
& f_e \leq \mathbf{u}_s x_s && \forall\, e = (s, o^-_S) \in E^{S^-}_{\mathrm{ext}} \\
& f_{(r, o^-_r)} \leq \mathbf{u}_r && \\
& f_e \leq \mathbf{u}_e && \forall\, e \in E_G \\
& f_e = 1 && \forall\, e \in E^{T^+}_{\mathrm{ext}} \\
& f_e = x_s && \forall\, e = (o^+, s) \in E^{S^+}_{\mathrm{ext}} \\
& x_s \in \{0, 1\} && \forall\, s \in S \\
& f_e \in \mathbb{Z}_{\geq 0} && \forall\, e \in E_{\mathrm{ext}}
\end{aligned}
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$$

Connectivity Inequalities

STP Excursion [7]

$$
\text{(uSP)} \quad
\begin{array}{rll}
\min & c^T x & \\
(i) & x(\delta(W)) \geq 1, & \text{for all } W \subset V, W \cap T \neq \emptyset, \\
 & & (V \setminus W) \cap T \neq \emptyset, \\
(ii) & 0 \leq x_e \leq 1, & \text{for all } e \in E, \\
(iii) & x \text{ integer}, &
\end{array}
$$

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(iii) & x \text{ integer}, & &
\end{array}
$$

$\forall\ W \subseteq V_G, s \in W \cap S \neq \emptyset.\ \ f(\delta^+_{E^R_{\text{ext}}}(W)) \geq x_s$

'*From each processing node there exists a path towards the sender.*'

Exponentially many constraints, but ...

can be separated in polynomial time.

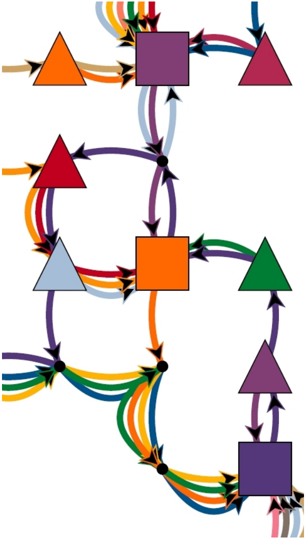
Decomposing flow is non-trivial (5 pages proof)!

Flow solution . . .

- contains cycles and
- represents *arbitrary* hierarchies.

Result

- Decomposition is *always* feasible
- Constructive proof:
  polynomial time algorithm



Combinatorial Heuristic: GreedySelect

Combinatorial Heuristics

On Chickens and Eggs

- How and when to place processing functionality?
- How and when to reserve bandwidth for routes?
- How to react to infeasibilities?

Our Approach

- Place processing functionality and reserve bandwidth jointly.
- Try to avoid infeasibilities by proactive routing decisions.

GreedySelect Heuristic

Greedily either . . .

- connect a single node to the connected component of the receiver or
- connect multiple nodes to an inactive processing node

minimizing the averaged discounted cost per connected node.

Selecting processing node + terminals + paths : $\mathcal{O}(|V| \cdot |E| + |V|^2 \log |V|)$

compute $\mathcal{P}_{\bar{s}} \triangleq (\bar{s} \in \bar{S}, T' \subseteq \bar{T}, \mathcal{P}_{T'} = \{P_{t,\bar{s}} | t \in T'\})$,

such that $P_{t,\bar{s}}$ connects t to $\bar{s}$,

$u^{\bar{s}}(e) - |\mathcal{P}_{T'}[e]| \geq 0$ for all $e \in E_G$,

$2 \leq |T'| \leq u_{r,S}(\bar{s})$

minimizing $c_{\bar{s},T'} \triangleq \left(\sum_{t \in T'} (c_E(P_{t,\bar{s}}) - c_E(P_{t,R})) + c_E(P_{\bar{s},R}) + c_S(\bar{s}) \right) / |T'|$

VirtuCast Based Heuristics

Overview VirtuCast Based Heuristics

FlowDecoRound

- based on (simple) flow decomposition and rounding

MultipleShot

- treats processing (node) variables as probabilities
- iteratively tries to construct a solution using a MST variant
- recomputes LP and iterates, if unsuccesful

Greedy Diving

- activates single *best* processing (node) iteratively, recomputes LP
- afterwards fixing of flow variables in a similar fashion
- complex fallback mechanisms

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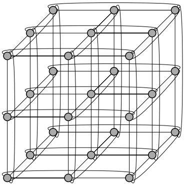
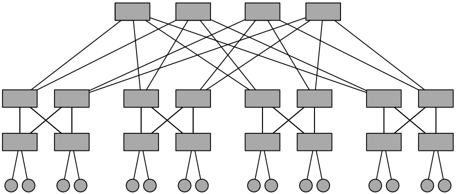
Computational Evaluation

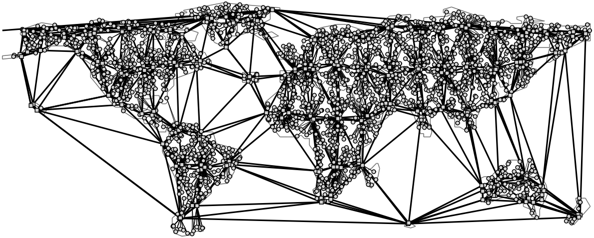
Topologies



3D torus



Fat tree



An ISP topology generated by IGen with 2400 nodes.

Computational Setup & Instances

Setup

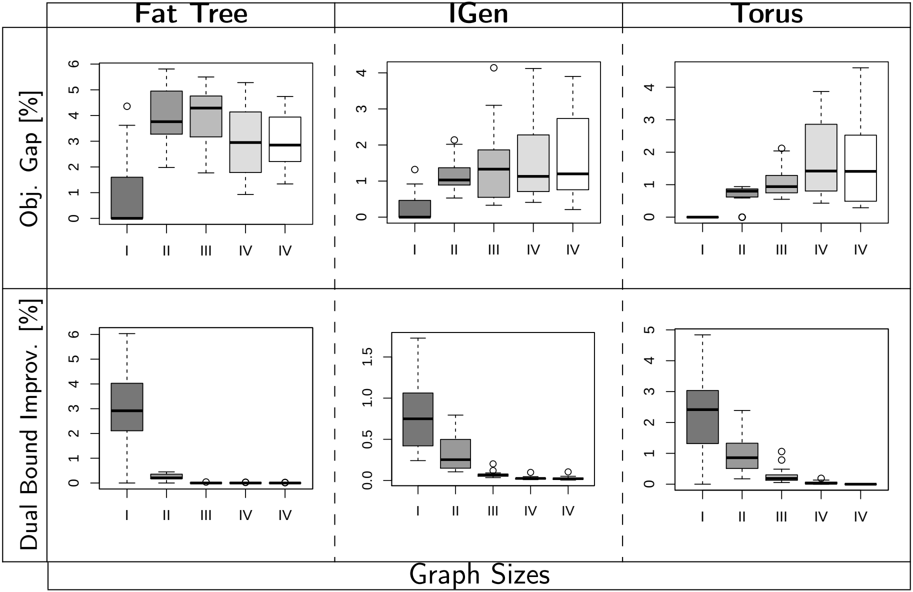
- 225 instances of five different graph sizes, costs, ...
- 1 hour runtime limit for computations
- All algorithms implemented in C/C++ using SCIP [1]

	Nodes	Edges	Processing Locations	Senders
Fat tree	1584	14680	720	864
3D torus	1728	10368	432	864
IGen	4000	16924	401	800

Table: Largest graph sizes

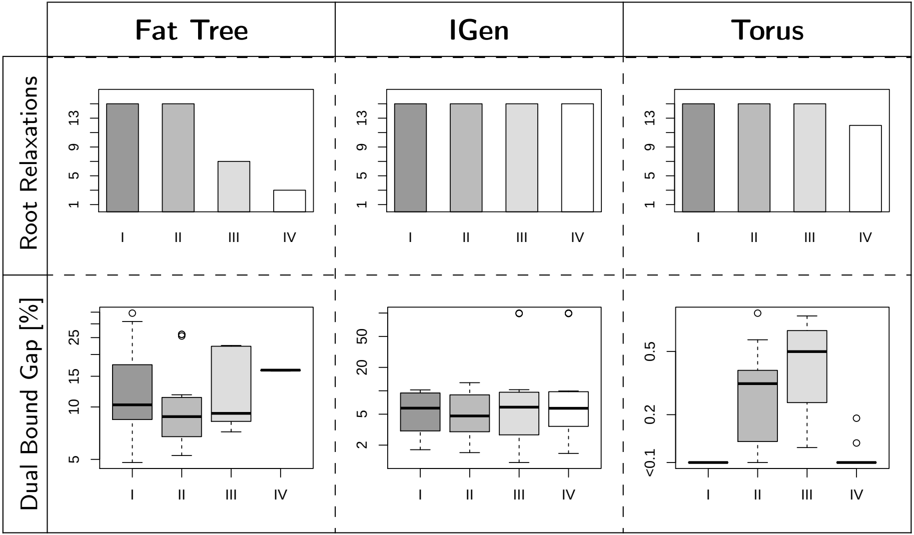
VirtuCast + LP-based Heuristics

VirtuCast + LP-based Heuristics



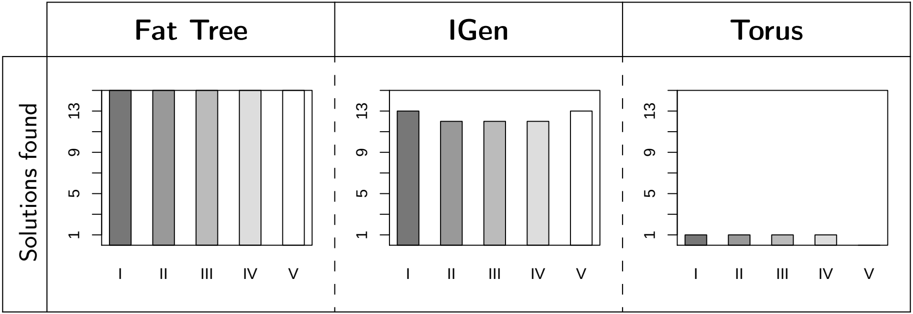
MCF-IP

MCF-IP: Performance

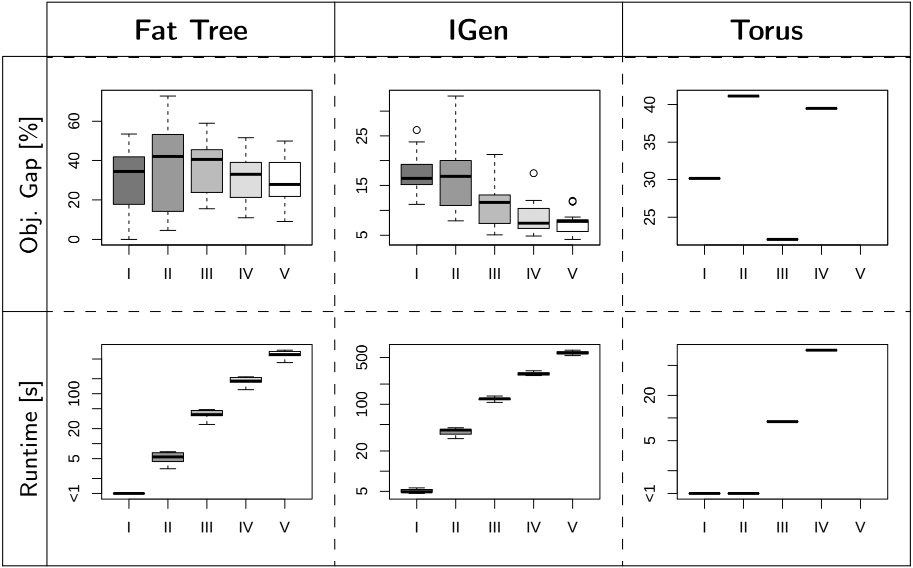


GreedySelect

GreedySelect: Efficacy

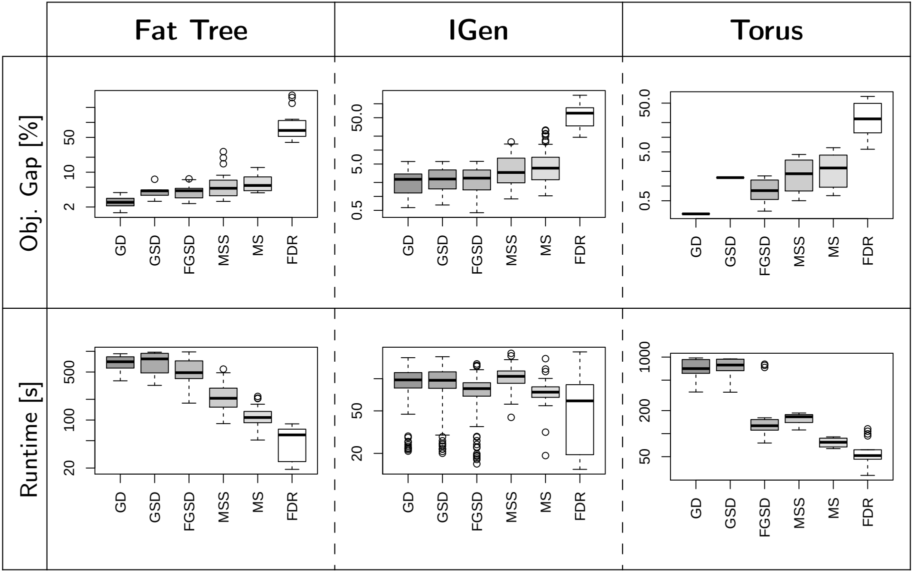


GreedySelect: Performance



LP-based Heuristics

LP-based Heuristics: Performance on graph size V



Conclusion

Most Important Results

VirtuCast Formulation

- *Single-commodity flow* IP formulation; considers only aggregate flow values
- Based on *novel* flow decomposition scheme
- *Enables* derivation of highly-efficient linear heuristics

IP formulation + linear heuristics

Highly efficient solver for CVSAP

Publications

Matthias Rost, Stefan Schmid: OPODIS 2013 & arXiv [14, 13]

Applications → Concise definition of CVSAP

Inapproximability

Approximations

- NVSTP
- VSTP
- VSAP

Exact Algorithms

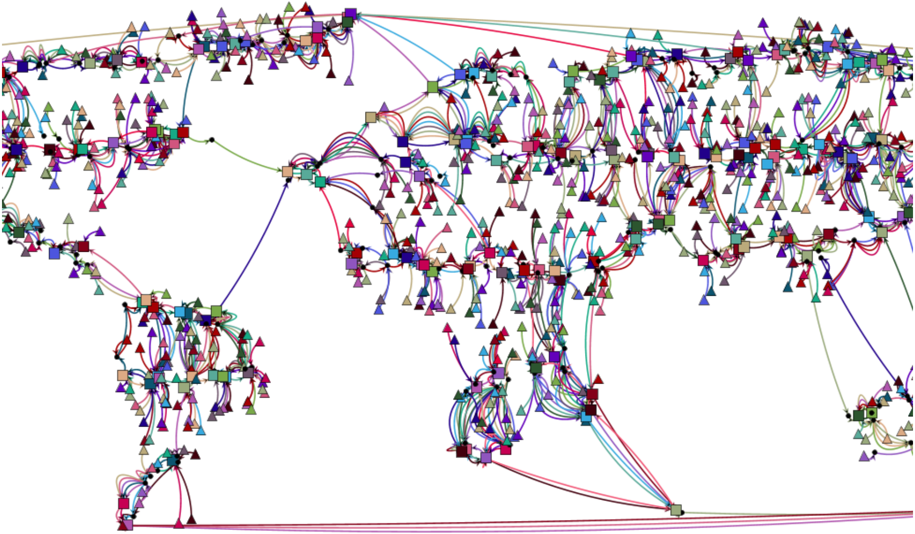
- multi-commodity flow
- single-commodity flow → VirtuCast

Heuristics

- FlowDecoRound
- MultipleShots
- GreedyDiving
- GreedySelect

Extensive explorative Computational Evaluation

Thanks



References I

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Related Work

Molnar: Constrained Spanning Tree Problems [9]

- Shows that optimal solution is a ‘spanning hierarchy’ and not a DAG.

Oliveira et. al: Flow Streaming Cache Placement Problem [11]

- Consider a weaker variant of multicasting CVSAP without bandwidth
- Give weak approximation algorithm

Shi: Scalability in Overlay Multicasting [15]

- Provided heuristic and showed improvement in scalability.

Future Work

Model Extensions

- prize-collecting variants
- concurrent multicast / aggregation sessions

Application Modeling

- Stratosphere II: Big Data
- UNIFY Project: flow analytics

IP formulation

- try to derive further cuts / facets

Future Work: UNIFY / Network Analytics

EU FP7 IP UNIFY [4]

- Considers *service chaining* in the wide-area network, connecting e.g. customers at home to (possibly multiple) datacenter
- Business perspective: SLAs must be guaranteed strictly, otherwise fines
  - KPIs need to be monitored constantly
  - Different measurements need to be collected the whole time

unifying cloud and carrier networks

Information Distribution

- Use multicast variant of CVSAP to distribute measurements.
- Placing processing nodes everywhere should be avoided due to the synchronization overhead (latencies).

Future Work: UNIFY / Network Analytics

EU FP7 IP UNIFY [4]

- Considers *service chaining* in the wide-area network, connecting e.g. customers at home to (possibly multiple) datacenter
- Business perspective: SLAs must be guaranteed strictly, otherwise fines
  - KPIs need to be monitored constantly
  - Different measurements need to be collected the whole time

unifying cloud and carrier networks

Information Aggregation

- Use aggregation variant of CVSAP to compute (subfunctions) of the KPIs on-the-fly
- Processing nodes may offer multicast functionality (see above) as well.

Backup

Integer Program 1: A-CVSAP-MCF

$$\text{minimize} \quad C_{\text{MCF}} = \sum_{e \in E_G} \mathbf{c}_e (f_e + \sum_{s \in S} f_{s,e}) + \sum_{s \in S} \mathbf{c}_s \cdot x_s \qquad \text{(MCF-OBJ)}$$

$$\text{subject to} \quad f^T(\delta^+_{E_{\text{MCF}}}(v)) = f^T(\delta^-_{E_{\text{MCF}}}(v)) + |\{v\} \cap T| \qquad \forall\, v \in V_G \qquad \text{(MCF-1)}$$

$$f^s(\delta^+_{E^s_{\text{MCF}}}(v)) = f^s(\delta^-_{E^s_{\text{MCF}}}(v)) + \delta_{s,v} \cdot x_s \qquad \forall\, s \in S, v \in V_G \qquad \text{(MCF-2)}$$

$$f^T_e + \sum_{s \in S} f^s_e \leq \begin{cases} \mathbf{u}_s x_s, & e = (s, \text{o}^-), s \in S \\ \mathbf{u}_r, & e = (r, \text{o}^-) \\ \mathbf{u}_e, & e \in E_G \end{cases} \qquad \forall e \in E_{\text{MCF}} \qquad \text{(MCF-3)}$$

$$-|S|(1 - f^s_{\bar{s},\text{o}^-}) \leq p_s - p_{\bar{s}} - 1 \qquad \forall\, s, \bar{s} \in S \qquad \text{(MCF-4)}$$

$$f^s_{(\bar{s},\text{o}^-)} \leq x_{\bar{s}} \qquad \forall\, s \in S, \bar{s} \in S - s \qquad \text{(MCF-5}^\star\text{)}$$

$$f^s_{s,\text{o}^-} = 0 \qquad \forall\, s \in S \qquad \text{(MCF-6}^\star\text{)}$$

$$f^s_{\bar{s},\text{o}^-} + f^{\bar{s}}_{s,\text{o}^-} \leq 1 \qquad \forall\, s, \bar{s} \in S \qquad \text{(MCF-7}^\star\text{)}$$

$$x_s \in \{0, 1\} \qquad \forall\, s \in S \qquad \text{(MCF-8)}$$

$$f^T_e \in \mathbb{Z}_{\geq 0} \qquad \forall\, e \in E_{\text{MCF}} \qquad \text{(MCF-9)}$$

$$f^s_e \in \{0, 1\} \qquad \forall\, s \in S, e \in E_{\text{MCF}} \qquad \text{(MCF-10)}$$

$$p \in [0, |S| - 1] \qquad \forall\, s \in S \qquad \text{(MCF-11)}$$

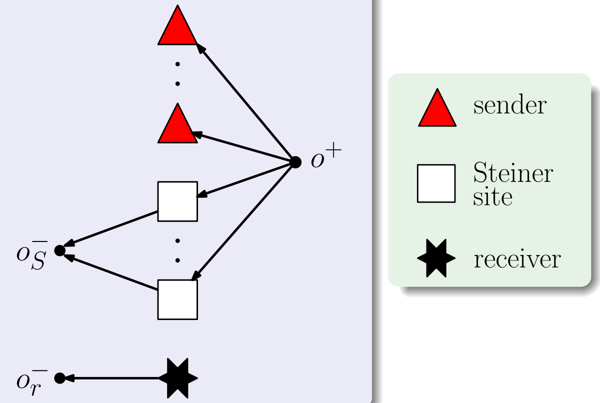
IP Formulation

Extended Graph

Additional nodes

- source o^+
- sinks o_r^- and o_S^-

Additional edges



IP Formulation I

$$
\begin{aligned}
\text{minimize} \quad & C_{\text{IP}}(x, f) = \sum_{e \in E_G} \mathbf{c}_e f_e + \sum_{s \in S} \mathbf{c}_s x_s \\
\text{subject to} \quad & f(\delta^+_{E_{\text{ext}}}(v)) = f(\delta^-_{E_{\text{ext}}}(v)) && \forall\, v \in V_G \\
& f(\delta^+_{E^R_{\text{ext}}}(W)) \geq x_s && \forall\, W \subseteq V_G, s \in W \cap S \neq \emptyset \\
& f_e = 1 && \forall\, e = (\text{o}^+, t) \in E^{T^+}_{\text{ext}} \\
& f_e = x_s && \forall\, e = (\text{o}^+, s) \in E^{S^+}_{\text{ext}} \\
& x_s \in \{0, 1\} && \forall\, s \in S \\
& f_e \in \mathbb{Z}_{\geq 0} && \forall\, e \in E_{\text{ext}}
\end{aligned}
$$

Connectivity Inequalities

STP Excursion [7]

$$
\text{(uSP)} \quad \begin{array}{rll}
\min & c^T x & \\
(i) & x(\delta(W)) \geq 1, & \text{for all } W \subset V, W \cap T \neq \emptyset, \\
 & & (V \setminus W) \cap T \neq \emptyset, \\
(ii) & 0 \leq x_e \leq 1, & \text{for all } e \in E, \\
(iii) & x \text{ integer}, &
\end{array}
$$

$\forall\ W \subseteq V_G, s \in W \cap S \neq \emptyset.\ \ f(\delta^+_{E^R_{\text{ext}}}(W)) \geq x_s$

'From each activated Steiner site, there exists a path towards o_r^-*.'*

Exponentially many constraints, but ...

can be separated in polynomial time.

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'*From each activated Steiner site, there exists a path towards* o^-_r.'

Exponentially many constraints, but ...

can be separated in polynomial time.

Complete Formulation

$$
\begin{aligned}
\text{minimize} \quad & C_{\text{IP}}(x, f) = \sum_{e \in E_G} \mathbf{c}_e f_e + \sum_{s \in S} \mathbf{c}_s x_s \\
\text{subject to} \quad & f(\delta^+_{E_{\text{ext}}}(v)) = f(\delta^-_{E_{\text{ext}}}(v)) && \forall\, v \in V_G \\
& f(\delta^+_{E^R_{\text{ext}}}(W)) \geq x_s && \forall\, W \subseteq V_G, s \in W \cap S \neq \emptyset \\
& f_e \leq \mathbf{u}_s x_s && \forall\, e = (s, \mathrm{o}^-_S) \in E^{S^-}_{\text{ext}} \\
& f_{(r, \mathrm{o}^-_r)} \leq \mathbf{u}_r \\
& f_e \leq \mathbf{u}_e && \forall\, e \in E_G \\
& f_e = 1 && \forall\, e \in E^{T^+}_{\text{ext}} \\
& f_e = x_s && \forall\, e = (\mathrm{o}^+, s) \in E^{S^+}_{\text{ext}} \\
& x_s \in \{0, 1\} && \forall\, s \in S \\
& f_e \in \mathbb{Z}_{\geq 0} && \forall\, e \in E_{\text{ext}}
\end{aligned}
$$

Outline of Decomposition Algorithm

Iterate

1. select a terminal t
2. construct path P from t towards o_r^- or o_S^-
3. remove one unit of flow along P
4. connect t to the second last node of P and remove t

After each iteration

Problem size reduced by one.

Outline of Decomposition Algorithm

Reduced problem must be feasible

Removing flow must not invalidate any connectivity inequalities.

Principle: Repair & Redirect

- decrease flow on path edge by edge
- if connectivity inequalities are violated

  repair increment flow on edge to remain feasible

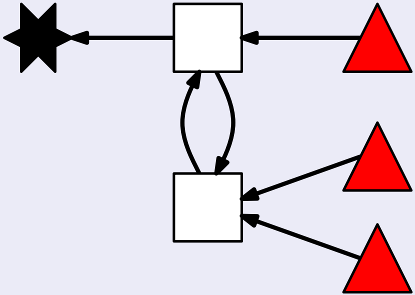
  redirect choose another path from the current node

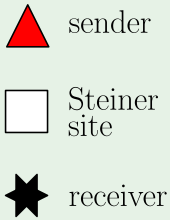
Theorem

Given an optimal solution, the Decomposition Algorithm computes a Virtual Arborescence in time $\mathcal{O}\left(|V_G|^2 \cdot |E_G| \cdot (|V_G| + |E_G|)\right)$.

Example

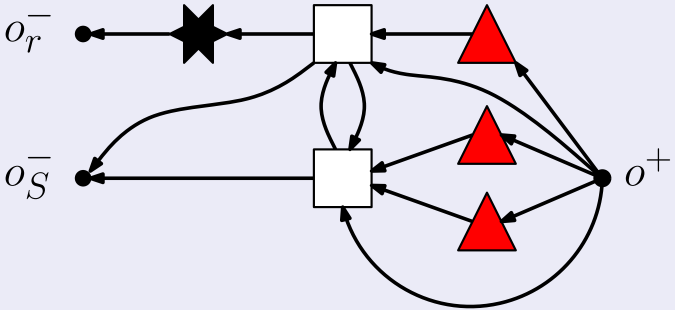
Scenario

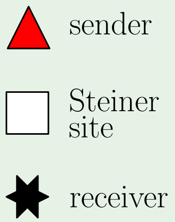




Example

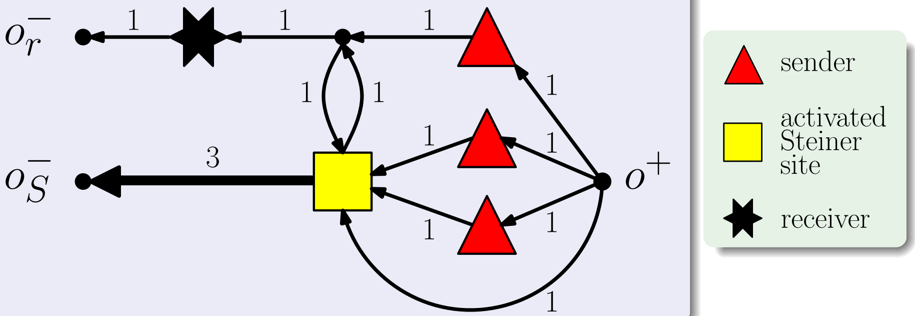
Extended Graph



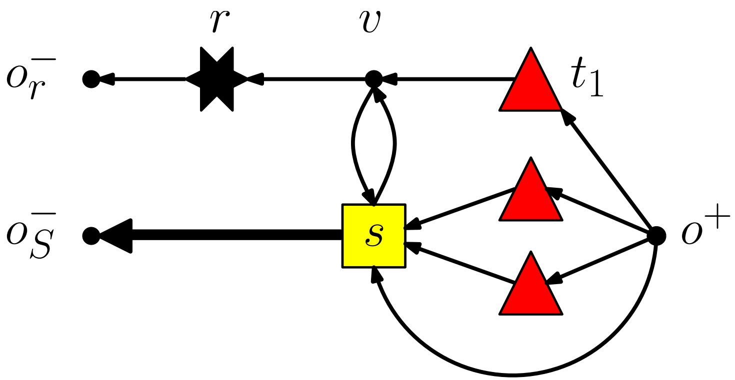


Example

Solution

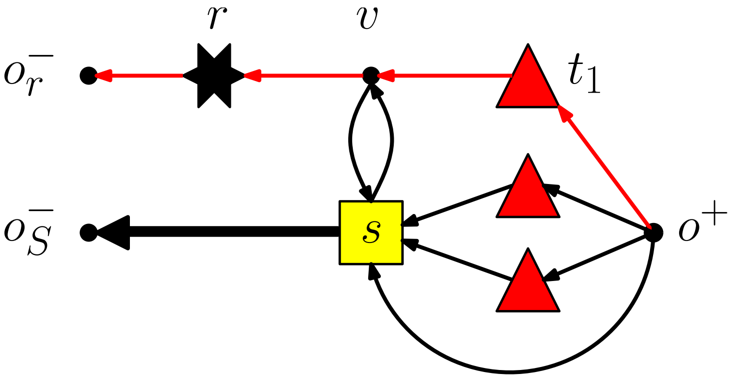


Decomposition Example I



Decomposition Example I

$$P = \langle o^+, t_1, v, r, o_r^- \rangle$$



Decomposition Example I

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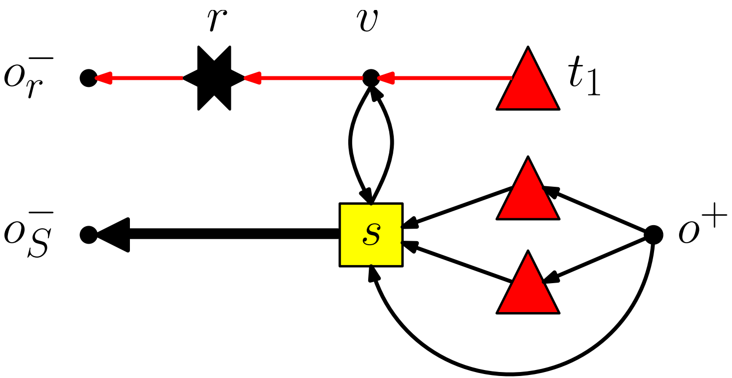

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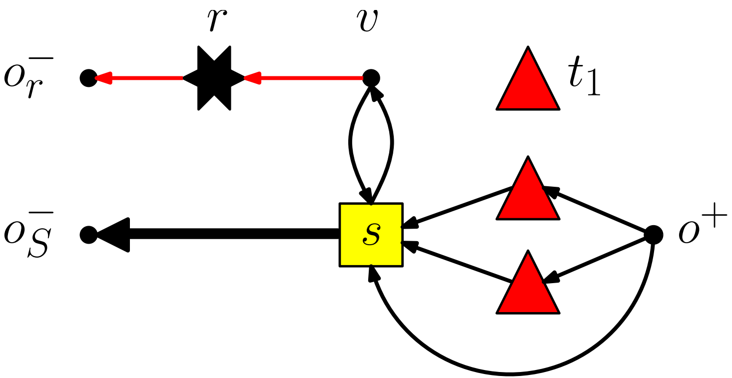

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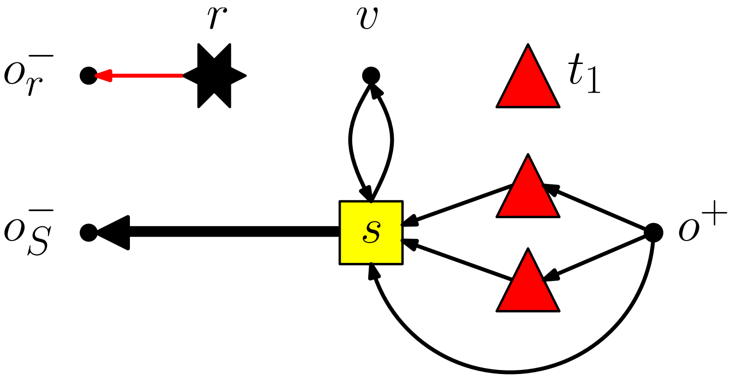
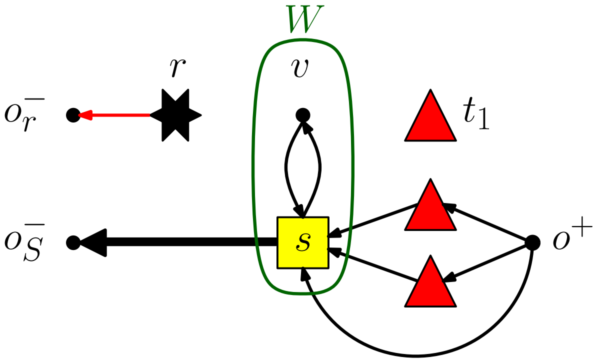


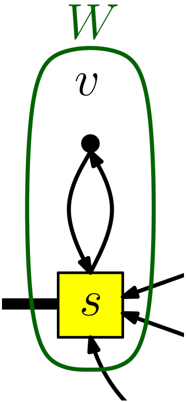
Redirecting Flow



Violation of Connectivity Inequality

$$f(\delta^+_{E^R_{\text{ext}}}(W)) \geq x_s \qquad \forall\ W \subseteq V_G, s \in W \cap S \neq \emptyset$$

Redirecting Flow



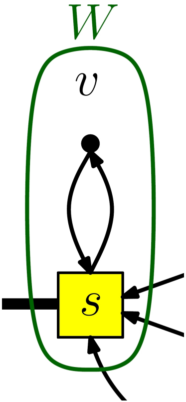
Redirection towards o_S^- is possible!

There exists a path from v towards o_S^- in W.

Reasoning

1. Flow preservation holds within W.
2. s could reach o_r^- via v before the reduction of flow.
3. v receives at least one unit of flow.
4. Flow leaving v must eventually terminate at o_S^-.

Redirecting Flow



Redirection towards o_S^- is possible!

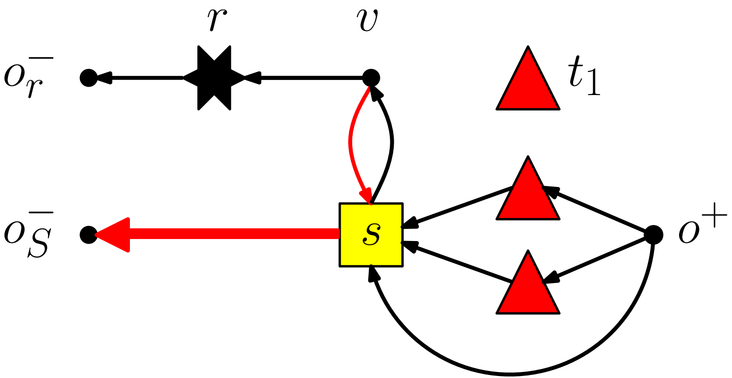
There exists a path from v towards o_S^- in W.

Reasoning

1. Flow preservation holds within W.
2. s could reach o_r^- via v before the reduction of flow.
3. v receives at least one unit of flow.
4. Flow leaving v must eventually terminate at o_S^-.

Decomposition Example II

$P = \langle o^+, t_1, v, s, o_S^- \rangle$



Decomposition Example II

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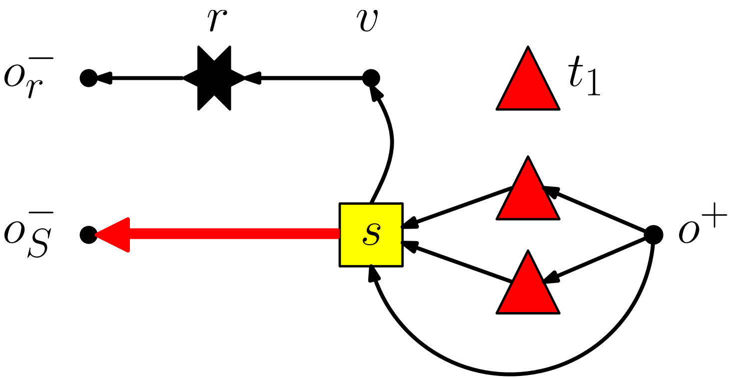
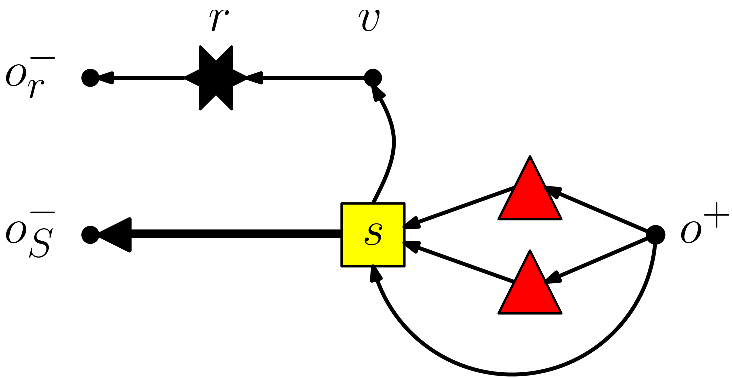
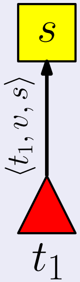


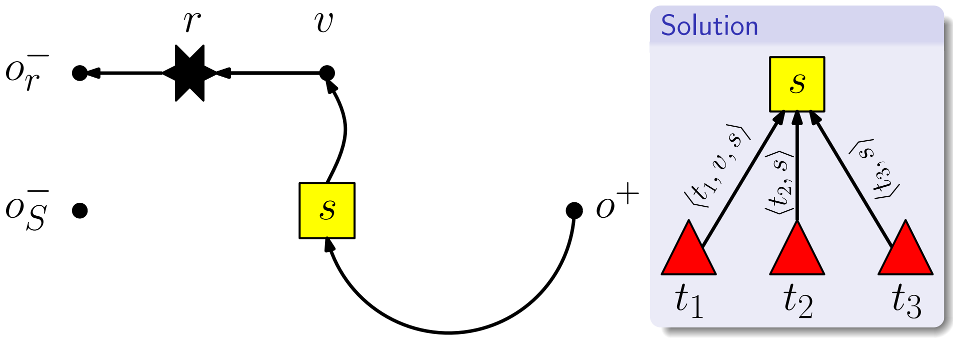
Decomposition Example II



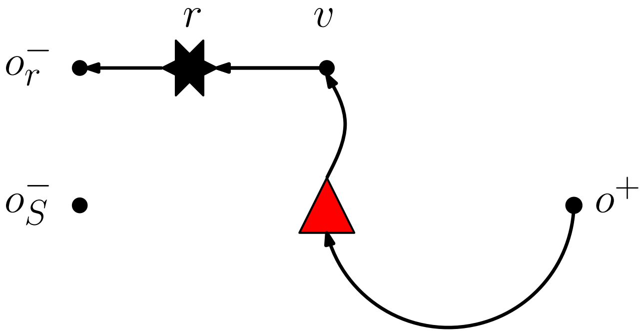
Solution

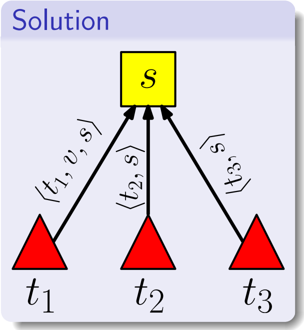


Decomposition Example II



Decomposition Example II





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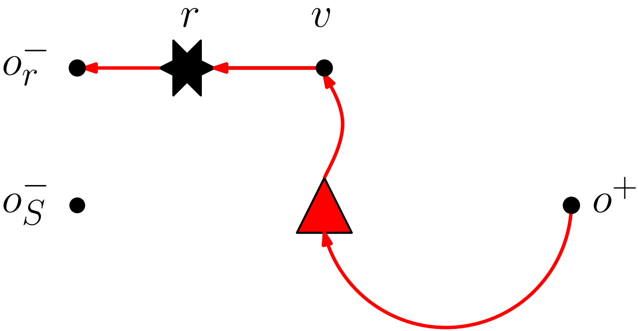
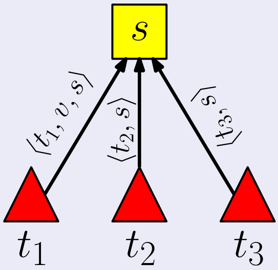
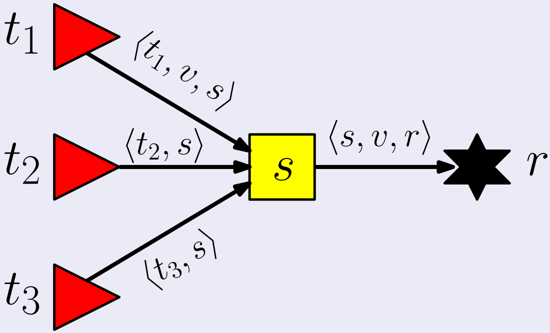


Solution



Decomposition Example II

Final Solution



Overview

Linear Relaxations

- The linear relaxation of an IP is obtained by relaxing the integrality constraints of the variables, thereby obtaining a Linear Program (LP).
- Solutions to linear relaxations are readily availabe when using branch-and-bound to solve an IP.
- May provide useful information to guide the construction of a solution.

Usage

- LP-based heuristics are employed within the VirtuCast *solver* to improve the bounding process.
- Yield polynomial time heuristics when used together with the root relaxation.

FlowDecoRound Heuristic

- computes a *flow* decomposition and connects nodes randomly according to the decomposition
- processing nodes are activated if another node node connects to it, must be connected itself
- failsafe: shortest paths

Algorithm 1: FlowDecoRound

Input : Network $G = (V_G, E_G, c_E, u_E)$, Request $R_G = (r, S, T, u_r, c_S, u_S)$, LP relaxation solution $(\hat{x}, \hat{f}) \in \mathcal{F}_{\text{LP}}$ to **??**

Output: A Feasible Virtual Arborescence $\hat{\mathcal{T}}_G$ or `null`

set $\hat{S} \triangleq \emptyset$ **and** $\hat{T} \triangleq \emptyset$ **and** $U = T$

set $\hat{V}_\mathcal{T} \triangleq \{r\},\ \hat{E}_\mathcal{T} \triangleq \emptyset$ **and** $\hat{\pi} : \hat{E}_\mathcal{T} \to \mathcal{P}_G$

set $u(e) \triangleq \begin{cases} u_E(e) & \text{, if } e \in E_G \\ u_r(r) & \text{, if } e = (r, \mathrm{o}_r^-) \\ u_S(s) & \text{, if } e = (s, \mathrm{o}_S^-) \in E_{\text{ext}}^{S-} \\ 1 & \text{, else} \end{cases}$ **for all** $e \in E_{\text{ext}}$

while $U \neq \emptyset$ **do**

  choose $t \in U$ uniformly at random **and set** $U \leftarrow U - t$

  set $\Gamma_t \triangleq$ `MinCostFlow` $\left(G_{\text{ext}}, \hat{f}, \hat{f}(\mathrm{o}^+, t), t, \{\mathrm{o}_S^-, \mathrm{o}_r^-\}\right)$

  set $\hat{f} \leftarrow \hat{f} - \sum_{(P,f) \in \Gamma_t, e \in P} f$

  set $\Gamma_t \leftarrow \Gamma_t \setminus \{(P,f) \in \Gamma_t | \exists e \in P . u(e) = 0\}$

  set $\Gamma_t \leftarrow \Gamma_t \setminus \{(P,f) \in \Gamma_t | (\hat{V}_\mathcal{T} + t, \hat{E}_\mathcal{T} + (t, P_{|P|-1}))$ is not acyclic $\}$

  if $\Gamma_t \neq \emptyset$ **then**

    choose $(P, f) \in \Gamma_t$ with probability $f / \left(\sum_{(P_j, f_j) \in \Gamma_t} f_j\right)$

    if $P_{|P|-1} \notin \hat{V}_\mathcal{T}$ **then**

      set $U \leftarrow U + P_{|P|-1}$ **and** $\hat{V}_\mathcal{T} \leftarrow \hat{V}_\mathcal{T} + P_{|P|-1}$

    set $\hat{V}_\mathcal{T} \leftarrow \hat{V}_\mathcal{T} + t$ **and** $\hat{E}_\mathcal{T} \leftarrow \hat{E}_\mathcal{T} + (t, P_{|P|-1})$ **and** $\hat{\pi}(t, P_{|P|-1}) \triangleq P$

    set $u(e) \leftarrow u(e) - 1$ **for all** $e \in P$

set $u(e) \leftarrow 0$ **for all** $e = (s, \mathrm{o}_S^-) \in E_{\text{ext}}^{S-}$ **with** $s \in S \wedge s \notin \hat{V}_\mathcal{T}$

set $\bar{T} \triangleq (T \setminus \hat{V}_\mathcal{T}) \cup (\{s \in S \cap \hat{V}_\mathcal{T} | \delta^+_{\hat{E}_\mathcal{T}}(s) = 0\})$

for $t \in \bar{T}$ **do**

  choose $P \leftarrow$ `ShortestPath` $\left(G_{\text{ext}}^u, c_E, t, \{\mathrm{o}_S^-, \mathrm{o}_r^-\}\right)$ **such that** $(\hat{V}_\mathcal{T} + t, \hat{E}_\mathcal{T} + (t, P_{|P|-1}))$ is acyclic

  if $P = \emptyset$ **then**

    return `null`

  set $\hat{V}_\mathcal{T} \leftarrow \hat{V}_\mathcal{T} + t$ **and** $\hat{E}_\mathcal{T} \leftarrow \hat{E}_\mathcal{T} + (t, P_{|P|-1})$ **and** $\hat{\pi}(t, P_{|P|-1}) \triangleq P$

  set $u(e) \leftarrow u(e) - 1$ **for all** $e \in P$

for $e \in \hat{E}_\mathcal{T}$ **do**

  set $P \triangleq \hat{\pi}(e)$

  set $\hat{\pi}(e) \leftarrow \langle P_1, \ldots, P_{|P|-1} \rangle$

set $\hat{\mathcal{T}}_G \triangleq$ Virtual Arborescence $(\hat{V}_\mathcal{T}, \hat{E}_\mathcal{T}, r, \hat{\pi})$

return `PruneSteinerNodes`$(\hat{\mathcal{T}}_G)$

Intermezzo: VCPrimConnect

Important Observation

If all placed processing nodes are already connected, all senders can be assigned *optimally* using a minimum cost flow.

Outline

1. connect all opened processing nodes in tree via a adaption of Prim's MST algorithm
2. assign all sending nodes using min-cost flow

Algorithm 2: VCPrimConnect

Input : Network $G = (V_G, E_G, c_E, u_E)$, Request $R_G = (r, S, T, u_r, c_S, u_S)$, *Partial* Virtual Arborescence $\mathcal{T}_G^P = (V_\mathcal{T}^P, E_\mathcal{T}^P, r, \pi^P)$

Output: Feasible Virtual Arborescence $\mathcal{T}_G = (V_\mathcal{T}, E_\mathcal{T}, r, \pi)$ or `null`

```
set U ≜ {v | v ∈ V_T^P \ {r}, δ^+_{E_T^P}(v) = 0}
set S̄ ≜ U ∩ S
set V_T ≜ V_T^P, E_T ≜ E_T^P and π(u,v) = π^P(u,v) for all (u,v) ∈ E_T
set u(e) ≜ u_E(e) − |π(E_T)[e]| for all e ∈ E_G
while S̄ ≠ ∅ do
    compute R ← {r' | r ∈ {r} ∪ (V_T ∩ S), r' reaches r in T_G, δ^-_{E_T}(r') < u_{r,S}(r')}
    compute P = MinAllShortestPath(G^u, c_E, S̄, R)
    if P = null then
        return null
    end
    set S̄ ← S̄ − P_1
    set E_T ← E_T + (P_1, P_{|P|}) and π(P_1, P_{|P|}) ≜ P
    set u(e) ← u(e) − 1 for all e ∈ P
end
set T̄ ≜ U ∩ T
set u_V(r') ≜ u_{r,S}(r') − δ^-_{E_T}(r') for all r' ∈ {r} ∪ (V_T ∩ S)
compute Γ = {P^t̄} ← MinCostAssignment(G, c_E, u, u_V, T̄, {r} ∪ V_T ∩ S)
if Γ = ∅ then
    return null
end
set E_T ← E_T + (t, P^t_{|P^t|}) and π(t, P^t_{|P^t|}) ≜ P^t for all P^t ∈ Γ
return T_G ≜ (V_T, E_T, r, π)
```

MultipleShots

- treats node variables as probabilities and iteratively places processing functionality accordingly
- tries to generate a feasible solution in each round via VCPrimConnect

Algorithm 3: MultipleShots

Input : Network $G = (V_G, E_G, c_E, u_E)$, Request $R_G = (r, S, T, u_r, c_S, u_S)$, LP relaxation solution $(\hat{x}, \hat{f}) \in \mathcal{F}_{\text{LP}}$ to ??

Output: A Feasible Virtual Arborescence $\hat{\mathcal{T}}_G$ or `null`

set $\lfloor S \rfloor \triangleq \{s \in S | \hat{x}_s \leq 0.01\}$ **and** $\lceil S \rceil \triangleq \{s \in S | \hat{x}_s \geq 0.99\}$
`addConstraintsLocally`($\{x_s = 0 | s \in \lfloor S \rfloor\} \cup \{x_s = 1 | s \in \lceil S \rceil\}$)
set $\dot{S}_0 \triangleq \lfloor S \rfloor \cup$ **and** $\dot{S}_1 \triangleq \lceil S \rceil$
`disableGlobalPrimalBound()`
repeat
  $(\hat{x}, \hat{f}) \leftarrow$ `solveSeparateSolve()`
  if `infeasibleLP()` **return** *null*
  set $\lfloor S \rfloor \triangleq \{s \in S | \hat{x}_s \leq 0.01\}$ **and** $\lceil S \rceil \triangleq \{s \in S | \hat{x}_s \geq 0.99\}$
  `addConstraintsLocally`($\{x_s = 0 | s \in \lfloor S \rfloor\} \cup \{x_s = 1 | s \in \lceil S \rceil\}$)
  set $\dot{S}_0 \leftarrow \dot{S}_0 \cup \lfloor S \rfloor$ **and** $\dot{S}_1 \leftarrow \dot{S}_1 \cup \lceil S \rceil$
  set $\hat{S} \triangleq S \setminus (\dot{S}_0 \cup \dot{S}_1)$
  if $\hat{S} \neq \emptyset$ **then**
    repeat
      set $S_1 \triangleq \hat{S}$
      remove s **from** S_1 **with probability** $1 - \hat{x}_s$ **for all** $s \in S_1$
      if $S_1 = \emptyset$ ***and*** $|S \setminus (\dot{S}_0 \cup \dot{S}_1)| < 10$ **then**
        set $S_1 \leftarrow S \setminus (\dot{S}_0 \cup \dot{S}_1)$
    until $S_1 \neq \emptyset$
    `addConstraintsLocally`($\{x_s = 1 | s \in S_1\}$)
    set $\dot{S}_1 \leftarrow \dot{S}_1 \cup S_1$
  $\hat{\mathcal{T}}_G^P \triangleq (\hat{V}_T^P, \hat{E}_T^P, r, \emptyset)$ **where** $\hat{V}_T^P \triangleq \{r\} \cup T \cup \dot{S}_1$ **and** $\hat{E}_T \triangleq \emptyset$
  set $\hat{\mathcal{T}}_G \triangleq$`VCPrimConnect`$(G, R_G, \hat{\mathcal{T}}_G^P)$
  if $\hat{\mathcal{T}}_G \neq$ *null* **then**
    return `PruneSteinerNodes`$(\hat{\mathcal{T}}_G)$
until $\dot{S}_0 \cup \dot{S}_1 = S$
return *null*

GreedyDiving

- aims at generating a feasible *IP* solution
- iteratively bounds at least a single variable from below, first fixing node variables
- complex failsafe: PartialDecompose + VCPrimConnect

Algorithm 4: GreedyDiving

Input : Network $G = (V_G, E_G, c_E, u_E)$, Request $R_G = (r, S, T, u_r, c_S, u_S)$, LP relaxation solution $(\hat{x}, \hat{f}) \in \mathcal{F}_{\mathrm{LP}}$ to ??

Output: A Feasible Virtual Arborescence $\hat{\mathcal{T}}_G$ or `null`

set $\lfloor S \rfloor \triangleq \{s \in S | \hat{x}_s \leq 0.01\}$ **and** $\lceil S \rceil \triangleq \{s \in S | \hat{x}_s \geq 0.99\}$
`addConstraintsLocally`$(\{x_s = 0 | s \in \lfloor S \rfloor\} \cup \{x_s = 1 | s \in \lceil S \rceil\})$
set $\dot{S} \triangleq \lfloor S \rfloor \cup \lceil S \rceil$ **and** $\dot{E} \triangleq \emptyset$
do
  $(\hat{x}', \hat{f}') \leftarrow$ `solveSeparateSolve()`
  if *infeasibleLP() and* $\dot{S} = S$ **then**
    break
  else if *infeasibleLP() or objectiveLimit()* **then**
    return *null*
  set $(\hat{x}, \hat{f}) \leftarrow (\hat{x}', \hat{f}')$
  if $\dot{S} \neq S$ **then**
    set $\lfloor S \rfloor \triangleq \{s \in S | \hat{x}_s \leq 0.01\}$ **and** $\lceil S \rceil \triangleq \{s \in S | \hat{x}_s \geq 0.99\}$
    `addConstraintsLocally`$(\{x_s = 0 | s \in \lfloor S \rfloor\} \cup \{x_s = 1 | s \in \lceil S \rceil\})$
    set $\dot{S} \leftarrow \dot{S} \cup \lfloor S \rfloor \cup \lceil S \rceil$
    set $\hat{S} \triangleq S \setminus \dot{S}$
    if $\hat{S} \neq \emptyset$ **then**
      choose $\hat{s} \in \hat{S}$ **with** $c_S(\hat{s}) / \hat{x}_{\hat{s}}$ **minimal**
      `addConstraintsLocally`$(\{x_{\hat{s}} = 1\})$
      set $\dot{S} \leftarrow \dot{S} + \hat{s}$
  else if $\dot{E} \neq E_{\mathrm{ext}}$ **then**
    set $\lfloor E \rfloor \triangleq \{e \in E_{\mathrm{ext}} | \, |\hat{f}_e - \lfloor \hat{f}_e \rfloor| \leq 0.001\}$, $\lceil E \rceil \triangleq \{e \in E_{\mathrm{ext}} | \, |\hat{f}_e - \lceil \hat{f}_e \rceil| \leq 0.001\}$
    `addConstraintsLocally`$(\{f_e = \lfloor \hat{f}_e \rfloor | e \in \lfloor E \rfloor\} \cup \{f_e = \lceil \hat{f}_e \rceil | e \in \lceil E \rceil\}$
    set $\dot{E} \leftarrow \dot{E} \cup \lfloor E \rfloor \cup \lceil E \rceil$
    set $\hat{E} \triangleq E_{\mathrm{ext}} \setminus \dot{E}$
    if $\hat{E} \neq \emptyset$ **then**
      choose $\hat{e} \in \hat{E}$ **with** $\lceil \hat{f}_{\hat{e}} \rceil - \hat{f}_{\hat{e}}$ **minimal**
      `addConstraintsLocally`$(\{\hat{f}_{\hat{e}} \geq \lceil \hat{f}_{\hat{e}} \rceil\})$
      set $\dot{E} \leftarrow \dot{E} + \hat{e}$
  else
    break
set $\hat{f}_e \leftarrow \lfloor \hat{f}_e \rfloor$ **for all** $e \in E_{\mathrm{ext}} \setminus \dot{E}$
set $\hat{\mathcal{T}}_G^P \leftarrow$ `PartialDecompose` $(G, R_G, (\hat{x}, \hat{f}))$
return `VCPrimConnect`$(G, R_G, \hat{\mathcal{T}}_G^P)$